Complete Segal Spaces as a model of Higher Categories

M.Sc. Thesis

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UNIVERSITÀ DI TRENTO

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Category Theory

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1. reflexivity, $[n] \to [n]$
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3. $([n] \to [m]) \to [p] = [n] \to ([m] \to [p])$

Why Higher Category?

Examples

Definition

For a pointed topological space (X, x), the **loop space** $\Omega_x X$ is defined as the set of all continuous maps,

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The way to make them groups is by moving to higher categories (A_∞ space).

Background


Higher Cate-
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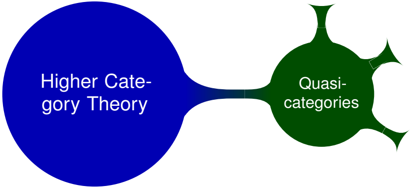
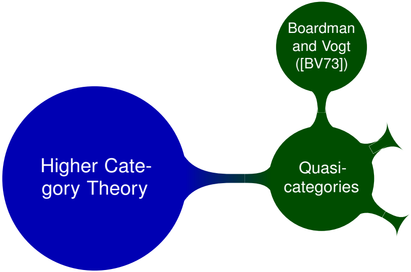

Higher Category Theory
Quasi-categories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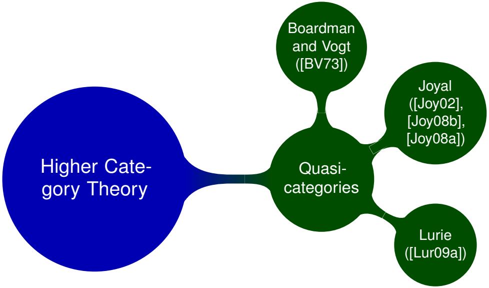
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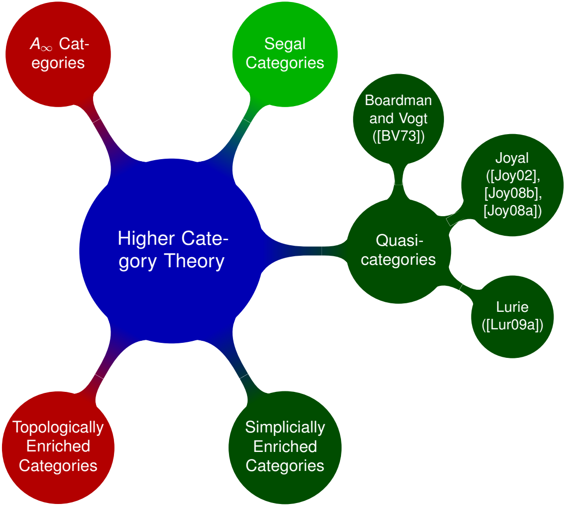
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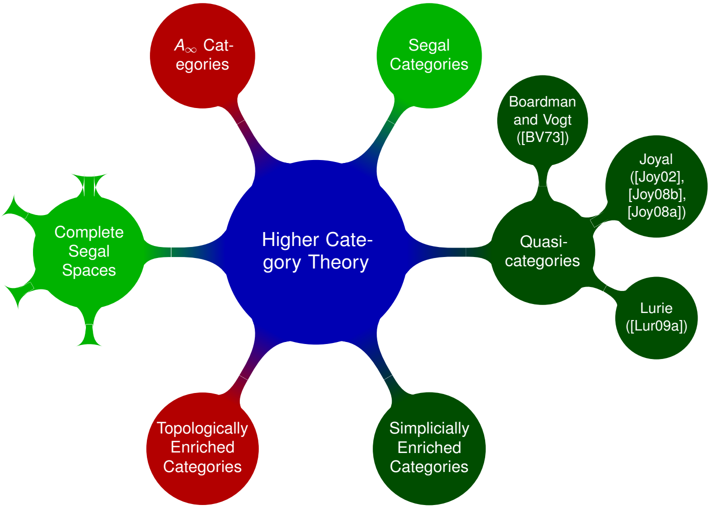
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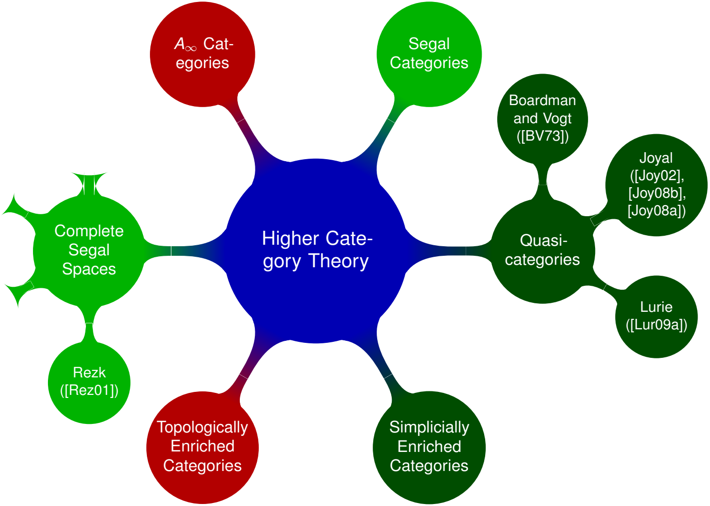
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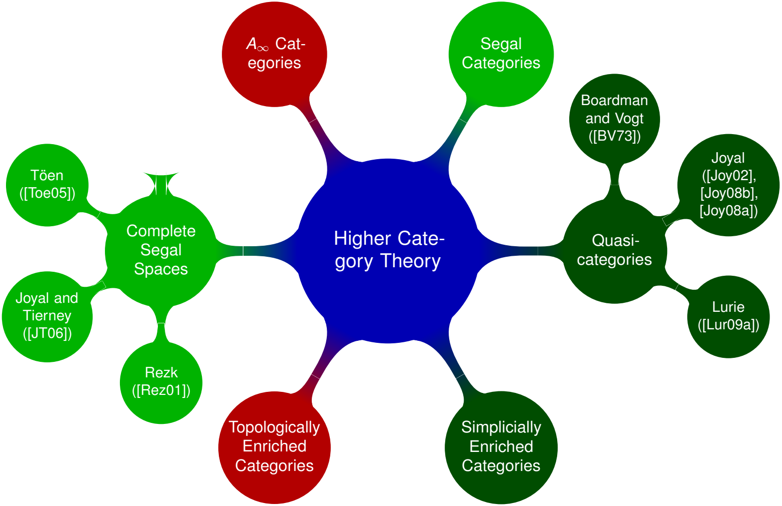
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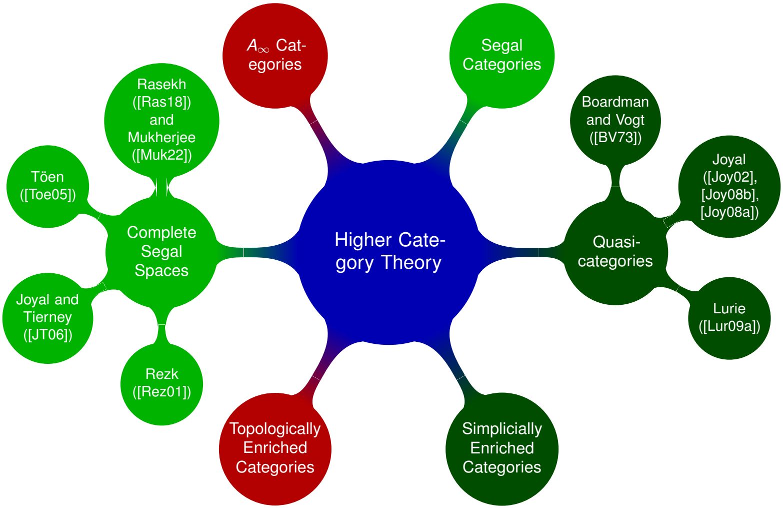
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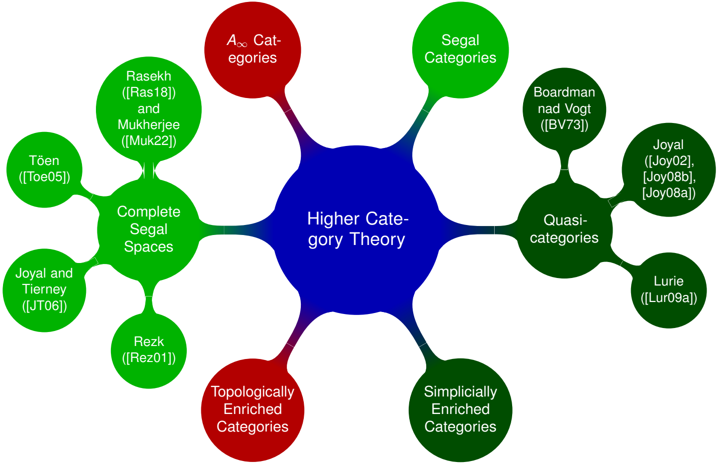

Conclusion

The category theory of complete Segal spaces has **not** been studied in details.
This thesis aims to fill this void!

Two Versions of Simplicial Sets

Definition

A **simplicial set** is a contravariant functor $X \colon \Delta^{op} \to Set$

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Note

By the Yoneda lemma, for any **sSet** X we have,

$$X_n \cong Hom_{\mathbf{sSet}}(\Delta^n, X)$$

Example

Let, $\mathcal{C}$ be a small category. Define the **nerve** of $\mathcal{C}$ to be the **sSet**, $N\mathcal{C}$ as follows:

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Version I

The nerve functor transforms any category into a **sSet.**

But what kind of **sSet** do we obtain from the nerve functor?

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Theorem ([Seg68])

Let X be a simplicial set that satisfies the Segal condition. Then there exists a category $\mathcal{C}$ such that X is equivalent to $N\mathcal{C}$.

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A simplicial set X satisfies the **Segal condition** if the map

$$X_n \xrightarrow{\cong} X_1 \underset{X_0}{\times} \dots \underset{X_0}{\times} X_1$$

is a bijection for $n \geq 2$.

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Lemma

The geometric realization of a simplicial set X is

$$|X| \cong \varinjlim_{\Delta^n \to X} |\Delta^n|$$

Definition

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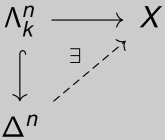
A simplicial set X is a **Kan complex** if every every horn in X has a filler,



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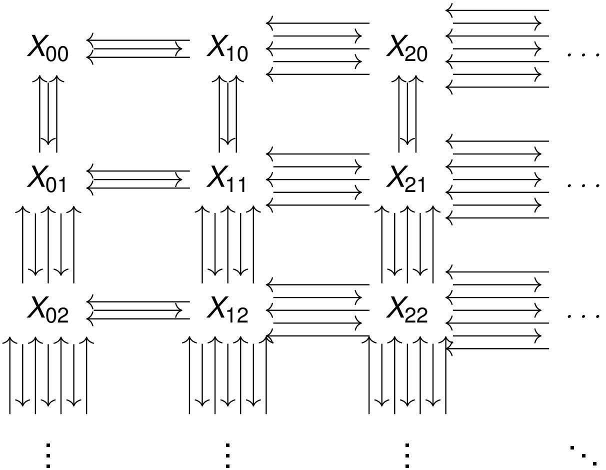

Note

$F(n)$ generates the columns and Δ^l generates the rows.

Model Categories

Reedy Fibrant Simplicial Space

Definition

A simplicial space X is called **Reedy fibrant** if $\forall n \geq 0$, the maps,

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Example

$F(n)$ is a Reedy fibrant simplicial space $\forall n \geq 0$.

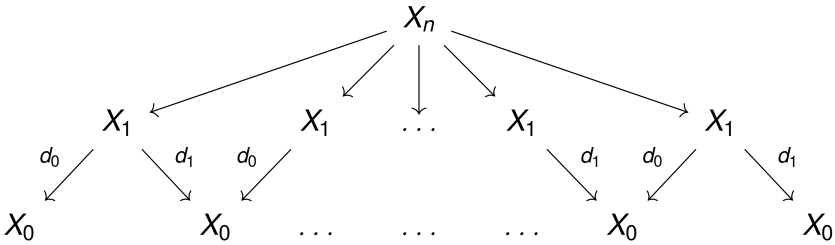
Segal Space

Definition

A Reedy fibrant simplicial space X is a **Segal space** if the maps,

$$X_n \xrightarrow{\simeq} X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1$$

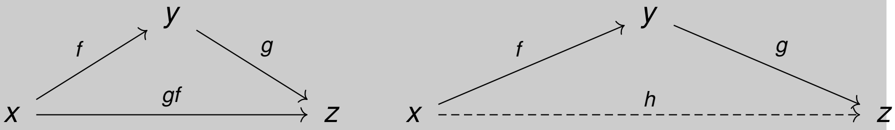
are Kan equivalences $\forall n \geq 2$.



Example

$$X_2 \xrightarrow{\simeq} X_1 \underset{X_0}{\times} X_1$$

$$map_X(x, y, z) \overset{\simeq}{\longmapsto} map_X(x, y) \times map_X(y, z)$$



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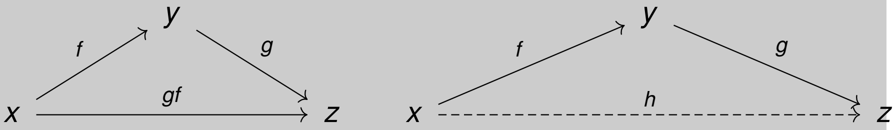

Note

The Segal condition does not guarantee uniqueness but only existence.

Composition

Definition

$$\begin{array}{ccccc} Comp(f,g) & \hookrightarrow & map_X(x,y,z) & \xrightarrow{d_1} & map_X(x,z) \\ \downarrow \simeq & & \downarrow \simeq & & \\ \Delta^0 & \longrightarrow & map_X(x,y) \times map_X(y,z) & & \end{array}$$

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Previous Example

$$Comp(f,g) = \left\{ \sigma = \begin{array}{ccc} & y & \\ {}^{f}\nearrow & & \searrow^{g} \\ x & \xrightarrow{h=gf} & z \end{array} \in X_2 \mid d_0\sigma = g, d_2\sigma = f \right\}$$

Since, $Comp(f,g)$ is contractible $\implies h = gf$

Complete Segal Spaces

Example

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3. composition of HoX,

$$Hom_{HoX}(x, y) \times Hom_{HoX}(y, z) \to Hom_{HoX}(x, z)$$
$$([f], [g]) \mapsto [f \circ g]$$

Complete Segal Space

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Definition

A Segal space X is called a **complete Segal space** if the map,

$$s_0 : X_0 \to X_{hoequiv}$$

is an equivalence of spaces.

Twisted Arrow Construction

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To any category **C**, we can associate a **twisted arrow category**, $Tw(\mathbf{C})$, where,

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1. the **objects** are morphisms $C \xrightarrow{f} D$ in $\mathbf{C}$ for $C, D \in \mathcal{O}(\mathbf{C})$
2. the **morphisms** are commutative diagram $$\begin{array}{ccc} C & \xrightarrow{k} & C' \\ f\downarrow & & \downarrow g \\ D & \xleftarrow[h]{} & D' \end{array}$$
3. and **composition of mophisms** $$\begin{array}{ccc} C & \xrightarrow{k'} & C' \\ f\downarrow & & \downarrow f' \\ D & \xleftarrow[h']{} & D' \end{array}$$ and $$\begin{array}{ccc} C' & \xrightarrow{k''} & C'' \\ f'\downarrow & & \downarrow f'' \\ D' & \xleftarrow[h'']{} & D'' \end{array}$$ are commutative diagram $$\begin{array}{ccc} C & \xrightarrow{k''\circ k'} & C'' \\ f\downarrow & & \downarrow f'' \\ D & \xleftarrow[h'\circ h'']{} & D'' \end{array}$$

Definition

If X is a quasi-category, then $Tw(X)$ is a simplicial set, i.e. explicitly,

$$Tw(X)_n = Hom_{\mathbf{sSet}}((\Delta^n)^{op} * \Delta^n, X) \cong X_{2n+1}$$

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Definition

If X is a simplicial space, $Tw(X)_{mn} := X_{2m+1,n}$, i.e. concretely,

$$Tw(F(m)) = F(2m+1)$$
$$Tw(\Delta^n) = \Delta^n$$

Main Results

Theorem

If X is a complete Segal space, then $\mathrm{Tw}(X)$ is a complete Segal space.

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1. $Tw(X)$ is Reedy fibrant
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3. $Tw(X)$ is a complete Segal space

Theorem

The projection map $Tw(X) \to X^{op} \times X$ is a left fibration.

Lemma

If X is a Reedy fibrant simplicial space, then $\mathrm{Tw}(X)$ is also a Reedy fibrant simplicial space.

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If X is a Reedy fibrant simplicial space, then $Tw(X)$ is also a Reedy fibrant simplicial space.

Proof Idea:

We analyze $Map(\partial F(n), Tw(X))$ and describe it as a colimit of the space X_{2n-1} and X_{2n-3} to prove

$$Map(F(n), Tw(X)) \to Map(\partial F(n), Tw(X))$$

is a Kan fibration.

Lemma

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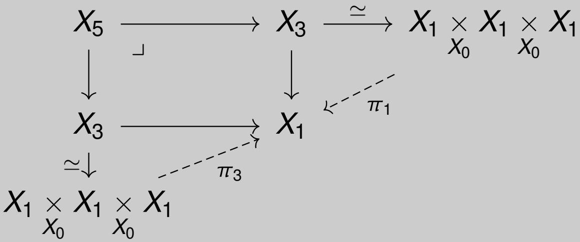
If X is a Segal space, then $Tw(X)$ is also a Segal space.

Proof Idea:

For $n = 2$,



we obtain $Tw(X)_2 \xrightarrow{\simeq} Tw(X)_1 \underset{Tw(X)_0}{\times} Tw(X)_1$.

Lemma

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- Pullback squares,

$$\begin{array}{ccccc} Tw(X)_0 & \longrightarrow & Tw(X)_{hoequiv} & \hookrightarrow & Tw(X)_1 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ X_0^{op} \times X_0 & \xrightarrow{\simeq} & X_{hoequiv}^{op} \times X_{hoequiv} & \hookrightarrow & X_1^{op} \times X_1 \end{array}$$

we obtain, $Tw(X)_0 \xrightarrow{\simeq} Tw(X)_{hoequiv}$ is an equivalence of spaces.

Theorem

The projection map $Tw(X) \to X^{op} \times X$ is a left fibration.

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Proof Idea:

- The map $Tw(X) \to X^{op} \times X$ is a Reedy fibration.

Theorem

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Proof Idea:

- The map $Tw(X) \to X^{op} \times X$ is a Reedy fibration.
- If X is a Segal space then the following diagram is a homotopy pullback square,

$$\begin{array}{ccc} Tw(X)_1 & \longrightarrow & Tw(X)_0 \\ \downarrow & \lrcorner & \downarrow \\ X_1^{op} \times X_1 & \longrightarrow & X_0^{op} \times X_0 \end{array}$$

References I

John. M. Boardman and R. M. Vogt.
Homotopy invariant algebraic structures on topological spaces.
Springer, 1973.

André Joyal and Myles Tierney.
Quasi-categories vs segal spaces.
Contemp. Math., 431, 08 2006.

Bertrand Toen.
Vers une axiomatisation de la théorie des catégories supérieures.
K-theory, 34:233–263, 2005.

Jacob Lurie.
On the classification of topological field theories.
Current Developments in Mathematics, 2008.

References II

Graeme Segal.
Classifying spaces and spectral sequences.
Publications Mathématiques de l'IHÉS, vol. 34, pp. 105–112, 1968.

Charles Rezk.
A model for the homotopy theory of homotopy theory.
Trans. Amer. Math. Soc. 353, 973-1007, 2001.

Andre Joyal.
Quasi-categories and kan complexes.
Journal of Pure and Applied Algebra, 175(1-3):207–222, nov 2002.

Jacob Lurie.
Higher topos theory.
Princeton University Press, 2009.

References III

Emily Riehl and Dominic Verity.
Elements of ∞-Category Theory.
Cambridge University Press, 2022.

André Joyal.
The theory of quasi-categories and its applications, 2008.
https://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern45-2.pdf

André Joyal.
Notes on quasi-categories, 2008.
http://www.math.uchicago.edu/~may/IMA/Joyal.pdf

Julia Bergner.
A Survey of $(\infty, 1)$-Categoriess, 2006.
https://arxiv.org/abs/math/0610239

References IV

- Julia Bergner.
  Equivalence of models for equivariant (∞, 1)-categories, 2014.
  https://arxiv.org/abs/1408.0038
- Nima Rasekh.
  Introduction to complete segal spaces, 2018.
  https://arxiv.org/abs/1805.03131
- Chirantan Mukherjee.
  Twisted Arrow Construction for Segal Spaces, 2022.
  https://arxiv.org/abs/2203.01788